Sheath theory in hot two-ion electron plasma

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Thermal effects are discussed in a two-ion electron plasma and for a model applicable to the oscillating sheath theory. The differences between the fluid and kinetic models are pointed out, as well as the differences between the approximative and exact kinetic description. It is shown that the approximative kinetic description, first, can not describe the additional acoustic mode which naturally exists in the plasma with an additional ion population with a finite temperature, and, second, it yields an inaccurate Landau damping of the bulk ion acoustic mode. The reasons for these two failures of the model are described. In addition to this, a fluid model is presented that is capable of capturing both of these features that are missing in the approximative kinetic description, i.e., two (fast and slow) ion acoustic modes, and the corresponding Landau damping of both modes.

1. Introduction

The behavior of the ion acoustic mode in the presence of an additional ion species may be drastically different compared to its behavior in a single ion plasma [1-4]. In particular, the Landau damping is increased with the addition of a lighter ion specie, which implies the possibility for controlling the mode behavior. Physically, this is equivalent to lowering the electron temperature and reducing the phase velocity, so that the damping is increased [1]. Experimental observations of those phenomena are available in the classic Refs. [2, 3] and more recently in Ref. [4]. We observe that these effects of the additional ions are just the opposite of the effects in a plasma with two (hot and cold) electron species [5, 6].

Each additional ion species introduces a new branch of acoustic oscillations in the system, complemented by various novel physical effects. However, in the context of the sheath theory, this domain of behavior of thermal ions has not yet been properly explored. This is the subject of the present work. In the case of a plasma with two ion species, one obtains two acoustic modes: one fast and one slow mode. As a result, the commonly used terms like the 'system sound velocity' appear to be redundant and may become inapplicable in most plasmas.

2. Cold ions

The dispersion equation for the ion acoustic (IA) mode propagating in a three component plasma consisting of electrons, and two cold ion species denoted by a and b is:

$$1+\frac{1}{k^2}=\frac{1}{[\omega-kv_{a0}(n_{e0}/n_{a0})^{1/2}]^2}+\frac{(n_{b0}/n_{a0})(m_a/m_b)}{[\omega-kv_{b0}(n_{e0}/n_{a0})^{1/2}]^2}. \quad (1)$$

Here, $k\equiv k\lambda_{de}$, ω is normalized to ω_{pa}, $v_{j0}\equiv v_{j0}/c_{sa}$ denotes the directed sheath currents/velocities of the two ion species, $j=a,b$, and $c_{sa}^2=\kappa T_e/m_a$.

Eq. (1) for cold ions is solved numerically. First, we set $v_{a0}=0.009$, $v_{b0}=0.011$, which should correspond to the situation deep inside the plasma, far from the sheath, and we also choose $m_a=10\,m_b$, and solve for $n_{b0}=0.2\,n_{a0}$ in terms of k. The result (for the ion acoustic mode) is presented in Fig. 1 with a dashed line, showing the tendency for the usual frequency saturation for large wave-numbers. Two additional complex-conjugate solutions for the sheath current driven modes, that also follow from Eq. (1), are about 2 orders of magnitude lower and are not of interest here.

3. Hot ions

Keeping the ion thermal effects, we re-derive the dispersion equation using the kinetic theory. It reads:

$$1+\sum_\alpha(\omega_{p\alpha}^2/k^2v_{T\alpha}^2)\left[1-\mathcal{Z}(\omega_{\alpha0}/kv_{T\alpha})\right]=0. \quad (2)$$

Here, $\alpha=e,a,b$, $\omega_{\alpha0}=\omega-kv_{\alpha0}$, and $\mathcal{Z}(x)=[x/(2\pi)^{1/2}]\int dy\exp(-y^2/2)/(x-y)$ is the plasma dispersion function, where $x\equiv\omega_{\alpha0}/(kv_{T\alpha})$, $y\equiv v/v_{T\alpha}$. For non-streaming electrons and in the limit

$$v_{Ta,b}\ll\omega/k\ll v_{Te}, \quad (3)$$

the standard expansions for $\mathcal{Z}$ are used, and the general dispersion equation (2) in that case becomes:

$$1+\frac{\omega_{pe}^2}{k^2v_{Te}^2}\left[1+i(\pi/2)^{1/2}\frac{\omega}{kv_{Te}}\right]$$
$$-\frac{\omega_{pa}^2}{k^2v_{Ta}^2}\left\{\frac{k^2v_{Ta}^2}{\omega_{a0}^2}+\frac{3k^4v_{Ta}^4}{\omega_{a0}^4}-i(\pi/2)^{1/2}\frac{\omega_{a0}}{kv_{Ta}}\right.$$
$$\left.\times\exp\left[-\omega_{a0}^2/(2k^2v_{Ta}^2)\right]\right\}-\frac{\omega_{pb}^2}{k^2v_{Tb}^2}\left\{\frac{k^2v_{Tb}^2}{\omega_{b0}^2}+\frac{3k^4v_{Tb}^4}{\omega_{b0}^4}\right.$$

$$-i(\pi/2)^{1/2}\frac{\omega_{b0}}{kv_{Tb}}\exp\left[-\omega_{b0}^2/(2k^2v_{Tb}^2)\right]\Big\} = 0. \quad (4)$$

We have solved Eq. (4) numerically by setting $T_a = T_b = T_e/15$, and for the same v_{j0}, n_{b0} as above. This result is also given in Fig. 1 (by full and dotted lines). The Landau damping has the maximum at $k \simeq 1$ where $|\gamma|/\omega \simeq 0.25$. Note that here and further in the text we present the *absolute* value of the Landau damping.

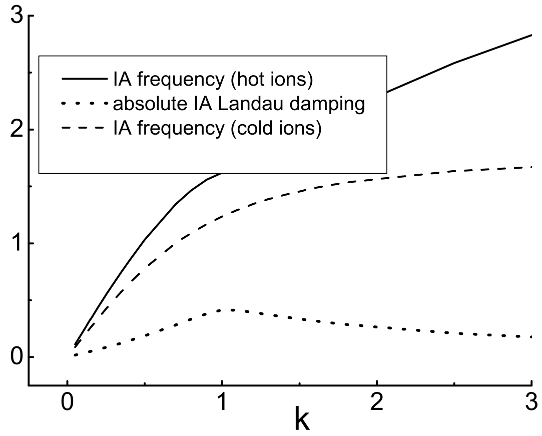

Figure 1. The positive solutions for the hybrid IA mode frequency (normalized to ω_{pa}) for hot (full line), and cold ions (dashed line), and the absolute value of the normalized Landau damping $|\gamma|$ (dotted line).

The presence of the second ion species affects both the mode frequency and the Landau damping, as shown in Fig. 2, where the hybrid IA mode frequency (normalized to ω_{pa}) is presented in terms of the number density of the species b, and for $T_a = T_b = T_e/30$, and $k = 0.1$ (cf. Fig 1). Here, in the beginning the Landau damping is increased by increasing the amount of the ion species b, but this goes only up to n_b/n_a of about 6 percent when $|\gamma|/\omega \simeq 0.11$. Note that for $n_{b0} = 0$ we have $|\gamma| = 2\cdot 10^{-4}$ (in units of ω_{pa}), i.e., an 80 times lower damping compared with the case when $n_{b0}/n_{a0} = 0.06$! Clearly, the presence of the additional ion species significantly affects the damping of the mode.

For hotter ions (i.e., $T_a = T_b = T_e/10$), the ratio $|\gamma|/\omega$ is increased, with the maximum $\simeq 0.22$ at $n_{b0}/n_{a0} = 0.2$. The Landau damping grows with the addition of more b-ions, yet this increase in the damping saturates for the number density n_b reaching about 20 percent. For $n_{b0} = 0$ it is $|\gamma| = 4\cdot 10^{-3}$, which is 12.5 lower compared with the case at $n_{b0}/n_{a0} = 0.2$. Hence, the second ion species may drastically increase the Landau damping of the hybrid IA mode, and we see also that its *relative* effect is more pronounced for lower ion temperatures.

4. Additional acoustic mode

The additional hot ion species imply additional branches of acoustic oscillations. This can only be seen

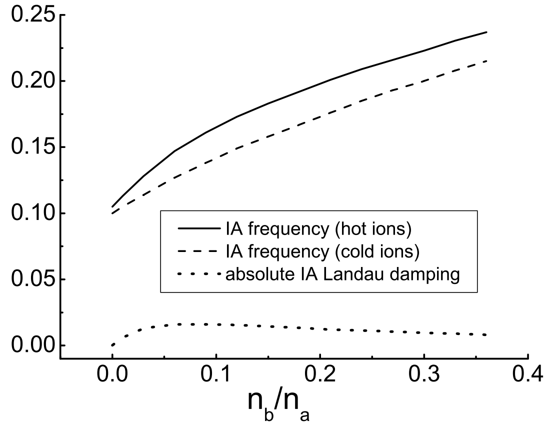

Figure 2. Full and dashed lines: the positive frequency (normalized to ω_{pa}) of the hybrid IA mode in terms of the number density of the second ion species, for $k = 0.1$. Dotted line: the absolute value of the normalized Landau damping $|\gamma|$.

by solving the general dispersion equation (2) numerically. Formally, the additional acoustic branches appear in the limit when the terms $k^2v_{Tj}^2$ are kept finite in comparison with ω^2. Therefore, in view of the condition (3) used in deriving Eq. (4), these additional acoustic branches are absent in the previous analysis and can not be deduced from Eq. (4). Hence, the approximative kinetic approach describes the Landau damping but it is unable to describe the *additional* acoustic branch. However, a fluid model easily captures both of these features within the fluid theory. For Boltzmannian electrons and with the help of the Poisson equation, the derivation of the dispersion equation is straightforward yielding [7]:

$$1 + \frac{1}{k^2\lambda_{de}^2} = \frac{\omega_{pa}^2}{\omega_{a0}\omega_{a1} - k^2v_{Ta}^2} + \frac{\omega_{pb}^2}{\omega_{b0}\omega_{b1} - k^2v_{Tb}^2}. \quad (5)$$

Here, $\omega_{j1} = \omega_{j0} + i\mu_{j0}k^2$. In the cold ions limit and for the given normalization, Eq. (5) becomes identical to Eq. (1). The term $\mu_{j0} \equiv \lambda v_{sj}/(2\pi^2 d_j)$, where $v_{sj}^2 = c_{sj}^2 + v_{Tj}^2$, follows from the ion momentum equation of the form

$$(\partial/\partial t + \vec{v}_{j0}\nabla)v_{j1} = -(q_j/m_j)\partial\phi_1/\partial x$$
$$-(\kappa T_j/m_j)\partial n_{j1}/\partial x + \mu_{j0}\partial^2 v_{j1}/\partial x^2,$$

where it has been introduced to describe the Landau damping. Such a fluid model has first been used in Ref. 8. It is convenient because it allows the use of the fluid theory, where the expressions can be analyzed more easily, as compared to the general kinetic expression containing the plasma dispersion function expressed through an integral. The term $d_j \equiv \delta_j/\lambda$ in μ_{j0} gives the ratio of the Landau attenuation length δ_j and wave-length λ. It is chosen in such a way that it is independent of the wavelength and the plasma density, and it depends on the ion temperature in a prescribed way. Such a fluid model for the intrinsically kinetic Landau damping is, first, *much more accurate* than the approximative kinetic Landau damping obtained after the expansion of $\mathcal{Z}(x)$ over the parameter x. Second, it gives *the same damping* as the exact kinetic Landau term. Third, it yields the analytical expression for the acoustic modes in two-ion electron plasmas with different temperatures of the two ion species [the alternative is numerically solving the general dispersion equation]. All these features follow after adopting the following expression:

$$d_j \equiv \delta_j/\lambda \approx 0.275708 + 0.0420737789\,\tau_j + 0.0890326\,\tau_j^2 - 0.011785\,\tau_j^3 + 0.0012186\,\tau_j^4. \quad (6)$$

Simple derivations for an electron-ion plasma yield the modeled fluid Landau damping as:

$$|\gamma_f|/\omega = 1/(2\pi d). \quad (7)$$

The approximative kinetic Landau damping is:

$$\frac{|\gamma_{app}|}{\omega} = \sqrt{\frac{\pi}{8}}\left[\left(\frac{m_e}{m_i}\right)^{1/2} + \tau(3+\tau)^{1/2}\exp\left(-\frac{3+\tau}{2}\right)\right]. \quad (8)$$

Finally, the exact kinetic Landau damping can be obtained only numerically. However, using the graph of the exact kinetic Landau damping from Ref. [5] one finds that it may be expressed by the following polynomial:

$$|\gamma_{ex}|/\omega = 0.681874545 - 0.36976364\tau + 0.09345889\tau^2 - 0.01203427\tau^3 + 0.0007523968\tau^4 - 0.000018\tau^5. \quad (9)$$

The above given statements about the accuracy of the expressions (7,8) can directly be checked by plotting the expressions (7-9) in terms of τ (details are given in Ref. [7]).

Using the same normalization as earlier, Eq. (5) becomes

$$1 + \frac{1}{k^2} = \Big\{ \left(\omega - kv_{a0}\sqrt{n_{e0}/n_{a0}}\right)\Big[\omega - kv_{a0}\sqrt{n_{e0}/n_{a0}} + i[kv_{sa}/(\pi d_a)]\sqrt{n_{e0}/n_{a0}}\Big] - k^2 T_a n_{e0}/(T_e n_{a0})\Big\}^{-1} + \frac{n_{b0}}{n_{a0}}\frac{m_a}{m_b}\Big\{\left(\omega - kv_{b0}\sqrt{n_{e0}/n_{a0}}\right)\Big[\omega - kv_{b0}\sqrt{n_{e0}/n_{a0}} + i[kv_{sb}/(\pi d_b)]\sqrt{n_{e0}/n_{a0}}\Big] - \frac{k^2 m_a n_{e0} T_b}{m_b n_{a0} T_e}\Big\}^{-1}. \quad (10)$$

Here, $v_{sj}^2 \equiv (c_{sj}^2 + v_{Tj}^2)/c_{sa}^2$. For comparison with previous cases, we solve Eq. (10) by taking the same parameters as in Fig. 1, i.e., $n_{b0} = 0.2n_{a0}$, $v_{a0} = 0.009$, $v_{b0} = 0.011$, $v_{sa} = 1.03$, $v_{sb} = 3.26$, $m_a = 10m_b$. We have also chosen $T_e = 11600$ K, $T_a = T_b = T_e/15$, and $m_b = 4m_p$ so that $c_{sa} = 1547$ m/s. The two acoustic modes are presented in Fig. 3. The given sheath currents make negligible frequency shifts in the two acoustic modes. The corresponding Landau damping of the two modes, presented in Fig. 4, shows an obvious difference as compared to Fig. 1. It is, first, lower by magnitude and, second, it shows no decrease for larger values of k. Hence, the decrease and saturation of the Landau damping, seen in previous figures, is clearly only due to the expansion of the plasma dispersion function, i.e., an artefact of the approximation used, and may not have any relevance to the real physical systems.

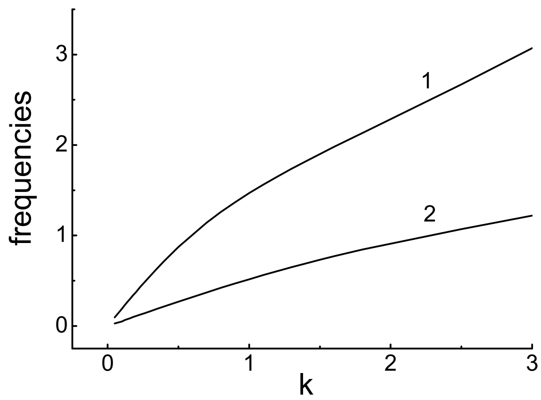

Figure 3. Two ion acoustic modes in two-ion electron plasma with $T_a = T_b = T_e/15$ (other parameters given in the text).

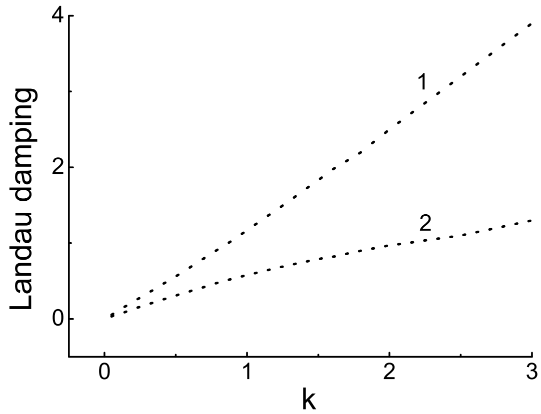

Figure 4. The absolute Landau damping $|\gamma| \equiv |\gamma|/\omega_{pa}$ (multiplied by 10^2) of the two IA modes from Fig. 3.

Similarly, when n_{b0} is increased the frequency and the Landau damping of the upper (fast) IA mode is increased. However, for the slow IA mode these both parameters are reduced when n_{b0} is increased. This is checked by setting $k = 0.3$ and for other parameters as above. The results are presented in Fig. 5 (for the two acoustic frequencies), and Fig. 6 for the absolute value of the Landau damping (multiplied by 10^3). This behavior is in agreement with results existing in the literature.

Setting the ion temperature to a larger value, i.e., $T_a = T_b = T_e/2$, the frequencies are increased by about a factor 1.5. However, the Landau damping is increased by a factor 100.

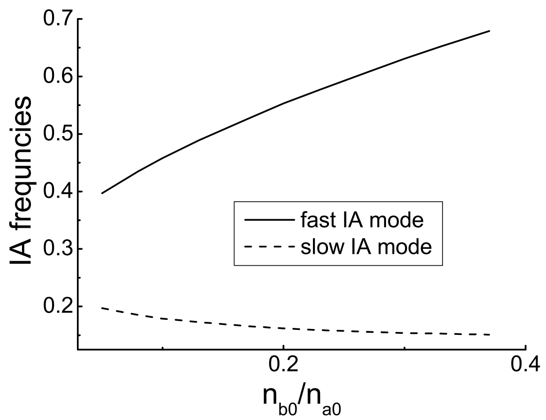

Figure 5. The frequencies of the two ion acoustic modes in terms of the number density of the ion species b for $T_a = T_b = T_e/15$.

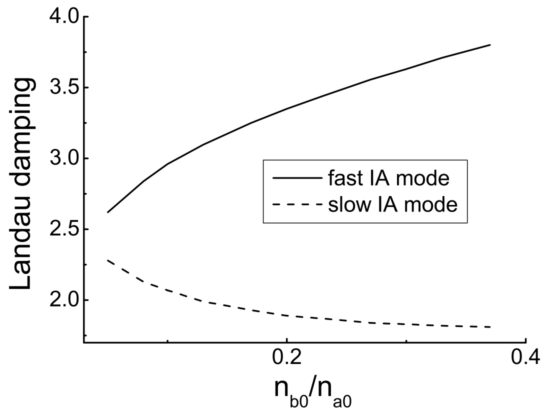

Figure 6. The absolute Landau damping $|\gamma| \equiv |\gamma|/\omega_{pa}$ (multiplied by 10^3) of the two IA modes from Fig. 5.

5. Summary

The ion thermal effects are investigated in the sheath theory in multi-component plasmas containing two ion species and electrons. Typically, this implies using the kinetic theory and we have demonstrated the differences between some results existing in the literature (obtained without ion thermal effects), and those obtained in the present analysis when the ion temperature effects are taken into account. The most important additional effect which follows from the finite ion temperature is the Landau damping, that is discussed in detail. However, the standard analysis implies the expansion of the plasma dispersion function, and as a result a) some phenomena related to ion thermal effects are missed in the procedure, and b) the mode behavior in some limits may show features that are not physical and that are only due to the mentioned expansion. The most obvious example of a) is the presence of an additional acoustic mode that can be seen only by solving the general kinetic dispersion equation (2) numerically. As for b), such an artefact of the expansion is the saturation and decrease of the Landau damping, as demonstrated in the text. In the case of hot ions, the expansion of the plasma dispersion function is not justified and an alternative approach is needed. We have shown that such a method exists within the framework of fluid theory. It is simple and suitable for analytical work, and at the same time is capable of capturing both effects, the additional ion acoustic mode and the Landau damping. In our earlier work [7] we have shown that, in fact, quantitatively it describes the Landau damping much more accurately than the analysis which follows after the expansion of the plasma dispersion function.

We conclude that more care is needed whenever the standard expansion of the plasma dispersion function is used. This may directly be seen by comparing Figs. 1, 2 on one side, and Figs. 4-6 on the other side. We stress also that even relatively small ion thermal effects imply the presence of an extra (slow) acoustic branch of ion oscillations, so that in fact two acoustic modes propagate in the plasma. The examples given here show that the frequencies of the two modes are not very distant even for relatively cool ions (in the given case the ion temperature of only about $1/15$ of the electron temperature), yet the two modes have rather different behavior. Therefore, terms like 'system sound velocity' may become inapplicable for most plasmas.

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